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Griffiths singularity in a non-dilute Ising chain

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Abstract. It is known that a one-dimensional random Ising model with a magnetic field h and two types of exchange bonds, $K_2 = \infty$ and $K_1 = 0$, has a Griffiths singularity in its magnetisation. Two arguments, one based on an expansion in the concentration of K_2 bonds and the other on decimation, are given which imply that a Griffiths singularity persists when $K_1 \neq 0$. The possibility of Griffiths singularities in non-dilute random magnets of higher dimension is briefly discussed.

1. Introduction

If an Ising model is randomly diluted, the magnetisation is a non-analytic function of the magnetic field h at $h = 0$ for temperatures below the critical temperature of the pure system (Griffiths 1969). Although such singularities probably are experimentally unobservable (Imry 1977), their unusual character continues to motivate theoretical interest (Wortis 1974, Domb 1974a, b, Harris 1975, Bakri and Stauffer 1976, Grinstein *et al* 1976); the spherical model has also been studied (Rauh 1976a, b, de Menezes *et al* 1977). The physical basis of the singularities is that the dilute system responds to a magnetic field as though arbitrarily large paramagnetic moments existed with finite probability. For instance, a one-dimensional Ising model with a concentration p of ferromagnetic bonds has the zero-temperature magnetisation (Matsubara *et al* 1973, Wortis 1974)

$$\bar{m} = (1-p)^2 \sum_{n=1}^{\infty} np^{n-1} \tanh(nh). \tag{1}$$

The poles of the giant-moment magnetisations $\tanh(nh)$ lie on the Im h axis and have a limit point at $h = 0$, where a Griffiths essential singularity occurs (Schwartz 1978).

In terms of the bond probability distribution

$$P(K) = (1-p)\delta(K-K_1) + p\delta(K-K_2), \tag{2}$$

the Griffiths singularities are known to exist at $K_1 = 0$ at sufficiently low temperature. It is natural to ask whether they continue to exist for $K_1 \neq 0$ or whether they are a peculiarity of the dilute system (for example, the limit point in equation (1) might be shifted to Im $h = \pm O(K_1^2)$ for small non-zero K_1). In this paper, this question will be investigated for the one-dimensional Ising model. For $p = 1$ the critical point is at

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$K_2=\infty$, and we shall search for signs of a Griffiths singularity when K_1 is finite and K_2 infinite. To our knowledge, the possibility of Griffiths singularities in non-dilute random systems has not been studied previously.

Although not rigorous, our arguments will indicate that Griffiths singularities persist when $K_1\neq 0$. This conclusion has been mentioned briefly elsewhere (Grinstein *et al* 1976).

The paper is divided into four parts. In § 2 we obtain the perturbation series in powers of p for the free energy, and examine the series for a Griffiths singularity. The expansion is divided into two series: the first gives (1) when $K_1=0$ and the second vanishes when $K_1=0$. Arguing that, at least for small K_1, singularities in the first series cannot be cancelled by singularities in the second, we find a Griffiths singularity at $h=0$ in the first series even when $K_1\neq 0$. In § 3 the Ising chain is studied by tracing over every spin in the system but one; a Griffiths singularity is found for $K_1\neq 0$. Also discussed is the possibility of Griffiths singularities in non-dilute random magnets of dimension $d>1$ for $K_1\neq 0$. Section 4 ends the paper with a summary and discussion of our arguments.

2. Griffiths singularity and perturbation theory

2.1. Perturbation theory

The quenched random spin-$\frac{1}{2}$ Ising model in a magnetic field h has the effective Hamiltonian

$$-\beta\mathcal{H}=\sum_{ij}K_{ij}s_is_j+h\sum_i s_i,\qquad s_i=\pm 1. \tag{3}$$

The quenched random bonds K_{ij} coupling each pair of adjacent spins are assigned according to the distribution (2). The free energy f per spin and magnetisation m per spin are given by

$$f=N^{-1}\ln\operatorname{tr}\mathrm{e}^{-\beta\mathcal{H}},\qquad m=\partial f/\partial h, \tag{4}$$

where N is the number of spins in the system. A bar superscript will be placed over quantities which are averaged over $P(K)$, e.g. $\bar{f}$, $\bar{m}$.

It is well known (Vedenov and Dykhne 1968, Lehman and McTague 1968, Fan and McCoy 1969, Chalupa *et al* 1976) that the thermodynamics of the random Ising chain can be expressed in terms of the solution of a linear integral equation. Defining

$$L_{K,h}(x)=\mathrm{e}^{-2h}(\mathrm{e}^{-2K}+x)/(1+\mathrm{e}^{-2K}) \tag{5a}$$

$$=\mathrm{e}^{-2h}x\quad(K=\infty), \tag{5b}$$

one obtains

$$\bar{f}=\bar{K}+h+\int_{-\infty}^{\infty}\mathrm{d}K\int_0^{\infty}\mathrm{d}R\,\nu(R)P(K)\ln[(1+\mathrm{e}^{-2K}R+\mathrm{e}^{-2(K+h)}+\mathrm{e}^{-2h}R)/(1+R)], \tag{6}$$

where $\nu(R)$ is the solution to

$$\nu(R)=\int_0^{\infty}\mathrm{d}R'\int_{-\infty}^{\infty}\mathrm{d}K\,P(K)\nu(R')\delta(R-L_{k,h}(R'))$$

$$\int_0^{\infty}\mathrm{d}R\,\nu(R)=1. \tag{7}$$

The properties of equation (7) are sufficiently complex that work continues to be done on the random Ising chain (Landau and Blume 1976, 1977, Fernández 1977, Morgenstern *et al* 1978, Vilenkin 1978).

As mentioned earlier, we are interested in the case $K_2=\infty$ because this is the critical point of the chain for $p=1$. For the dilute case $K_1=0$, equation (7) becomes an inhomogeneous equation which is easily solved by substituting the trial function $\nu(R)=\delta(R-R_0)$ and iterating. We find

$$\nu(R)=(1-p)\sum_{n=0}^{\infty}p^n\delta(R-\mathrm{e}^{-2(n+1)h}). \tag{8}$$

For $K_1\neq 0$ we get 2^n distinct delta-functions at the nth iteration, and the process cannot be completed analytically.

However, we can make progress by generating a formal expansion of $\nu(R)$ in powers of the concentration p of strong bonds; with such an expansion we can use equation (6) to calculate the thermodynamic functions in powers of p. We define

$$\nu(R)\equiv\sum_{n=0}^{\infty}p^n\nu_n(R), \tag{9}$$

where $\nu_n(R)$ is independent of p. Substituting (9) into (7) and equating coefficients of p, we obtain

$$\nu_{n+1}(R)=\nu_{n+1}(L_{K_1,h}(R))+\nu_n(L_{K_2,h}(R))-\nu_n(L_{K_1,h}(R))\qquad (n\neq 0) \tag{10a}$$

$$\nu_0(R)=\nu_0(L_{K_1,h}(R)). \tag{10b}$$

The functional equation (10a) for $\nu_{n+1}(R)$ may be solved by iteration if $\nu_n(R)$ is known. The ν_n's will be expressed in terms of the distribution function $\nu_0(R)$ for the pure Ising chain,

$$\nu_0(R)=\delta(R-R_0) \tag{11a}$$

$$R_0=L_{K_1,h}(R_0)=\mathrm{e}^{2K_1-h}[-\sinh h+(\sinh^2 h+\mathrm{e}^{-4K_1})^{1/2}] \tag{11b}$$

$$R_0=\mathrm{e}^{-2h}\quad (K_1=0). \tag{11c}$$

Our next task is to devise a compact notation to represent the functions $\nu_n(R)$. Since we are iterating bilinear transformations, it is natural to use continued fractions. Equation (10b) may be rewritten as

$$R_0=\mathrm{e}^{2(K_1-h)}-\mathrm{e}^{-2h}(\mathrm{e}^{4K_1}-1)/(\mathrm{e}^{2K_1}+R_0). \tag{12}$$

By iterating (12) repeatedly, we express R_0 as an infinite continued fraction. With respect to this fraction, we define $R(m_1,\ldots,m_n)$ with $m_i>0$, as the infinite continued fraction in which the m_ith terms of R_0 are replaced by

$$\begin{matrix}\cdot & & \\ & \cdot & \\ & & \cdot\ \mathrm{e}^{2(K_2-h)}-\dfrac{\mathrm{e}^{-2h}(\mathrm{e}^{4K_2}-1)}{\mathrm{e}^{2K_2}+\cdot}\\ & & \qquad\qquad\cdot\\ & & \qquad\qquad\quad\cdot\end{matrix} \tag{13}$$

To elucidate this definition, we give the result of iterating (10a) for ν_1 and ν_2:

$$\nu_1(R)=\sum_{m=1}^{\infty}[\delta(R-R(m))-\delta(R-R_0)] \tag{14a}$$

$$\nu_2(R)=\sum_{m_1,m_2=1}^{\infty}[\delta(R-R(m_2,m_1+m_2))-\delta(R-R(m_2))$$
$$-\delta(R-R(m_1+m_2))+\delta(R-R_0)]. \qquad (14b)$$

Now consider an ordered n-tuplet $(a_1, a_2, \ldots, a_n)$ with $a_i < a_{i+1}$. We define projection mappings P_k, with $k=0, 1, \ldots, n$, which project the set into a coordinate with $n-k$ terms and preserve the order relation among the elements. There are thus 2^n such mappings for $(a_1, a_2, \ldots, a_n)$. In this notation, the result of iterating the equation for $\nu_n(R)$ is

$$\nu_n(R)=\sum_{m_1,\ldots,m_n=1}^{\infty}\sum_{k=0}^{n}\sum_{\{P_k\}}(-)^{P_k}\delta$$
$$\times(R-R(P_k(m_n, m_n+m_{n-1}, \ldots, m_n+m_{n-1}+\ldots+m_1))). \qquad (15)$$

It is known that functions like $\nu(R)$ have unusual characteristics (Lieb and Mattis 1966). For our purposes $\nu(R)$ and $\nu_n(R)$ are useful only insofar as they yield a formal expansion of $\bar{f}$ and $\bar{m}$ in powers of p. As in many arguments to all orders in perturbation theory, formal manipulations were used to obtain the series (8) and (15), without regard for their convergence.

2.2. *Griffiths singularity*

We begin our search for the Griffiths singularity by re-establishing contact with the known results for the dilute chain. By splitting equation (15) into two pieces, we write

$$\nu(R)=\nu^{(1)}(R)+\nu^{(2)}(R)=\sum_{n=0}^{\infty}p^n[\nu_n^{(1)}(R)+\nu_n^{(2)}(R)] \qquad (16)$$

$$\nu^{(1)}(R)=\delta(R-R_0)+\sum_{n=1}^{\infty}p^n[\delta(R-R(1,\ldots,n))-\delta(R-R(1,\ldots,n-1))] \qquad (17a)$$

$$=(1-p)\sum_{n=0}^{\infty}p^n\delta(R-R(1,\ldots,n)). \qquad (17b)$$

Recalling ($5b$) and ($11c$), we obtain (8) from ($17b$) when $K_1=0$. From (12) and (13) we notice that the infinite continued fractions $R(m_1, \ldots, m_n)$ become finite when $K_1=0$. Substituting ($17b$) into the free energy (6), after a little algebra we recover the magnetisation (1). From (6) it is apparent that the poles of $\bar{m}$ are given by the roots of

$$1+R(1,\ldots,n)=0, \qquad (18)$$

which reduces to

$$1+e^{-2(n+1)h}=0, \qquad n=0, 1, \ldots \qquad (19)$$

for the dilute case, in agreement with Wortis (1974).

Inspection of (13) and (15) reveals that $\nu^{(2)}(R)=0$ when $K_1=0$ because all the terms of $\nu(R)$ not in $\nu^{(1)}$ cancel each other. If we ignore this cancellation, substitute $\nu^{(2)}$ into (6) and calculate $\bar{m}$, we find that at a given power of p^n in $\bar{m}$, no singularity from $\nu^{(2)}$ can cancel a singularity from $\nu^{(1)}$, i.e. no pole of $\bar{m}$ from $\nu^{(1)}$ has the same location and equal and opposite residue as any pole from $\nu^{(2)}$. Explicitly, the singularities from $\nu^{(1)}(R)$, for a given power of $n \geq 2$, are at the solutions of (cf (17*b*))

$$1+\mathrm{e}^{-2(m+1)h}=0, \qquad m=n, n-1. \tag{20}$$

The singularities from $\nu^{(2)}(R)$ are at

$$1+\mathrm{e}^{-2(m+1)h}=0, \qquad m=0, 1, \ldots, n-2. \tag{21}$$

Now consider $K_1 \neq 0$ and regard the $R(m_1, \ldots, m_n)$'s as Taylor series in K_1. In this case $\nu_2(R) \neq 0$. A little reflection on (6) convinces one that, as above for $K_1=0$, singularities from $\nu^{(2)}$ cannot exactly cancel singularities from $\nu^{(1)}$ contributing to $\bar{m}$. In other words, the singularities from the roots of (18) cannot be cancelled by other singularities. So if the roots of (18) lie on the Im h axis and have a limit point at $h=0$ for large n, our non-dilute Ising chain has a Griffiths singularity at $h=0$.

The solutions to (19) are (Wortis 1974)

$$h_{n0}=(m+\tfrac{1}{2})\pi\mathrm{i}/(n+1), \qquad m=\ldots, -1, 0, 1, \ldots. \tag{22}$$

For our purpose it is sufficient to study the effects of non-zero K_1 on the $m=0$ terms in (22). We rewrite (18) as

$$1+\mathrm{e}^{2K_1-(2n+1)h}[-\sinh h+(\sinh^2 h+\mathrm{e}^{-4K_1})^{1/2}]=0. \tag{23}$$

The solutions to (23) can be expanded formally around the values (22) in powers of K_1:

$$h_n=\sum_{m=0}^{\infty} h_{nm}K_1^m. \tag{24}$$

A simple computation yields

$$h_{n1}=(\tanh h_{n0})/(n+1). \tag{25}$$

Clearly h_{n1} becomes arbitrarily small as n becomes large. By substituting (24) into (23) and expanding in powers of K_1, we find that the factor of $\exp[-(2n+1)h]$ forces h_{nm} for any given m to vanish as n becomes sufficiently large. So when $K_1 \neq 0$, $h=0$ remains the limit point of solutions of (23), giving rise to a Griffiths singularity.

We can check this conclusion by studying the expansion of h_n in inverse powers of n:

$$h_n=\sum_{m=1}^{\infty} h'_{nm}(n+1)^{-m}. \tag{26}$$

A little algebra yields

$$h'_{n0}=(n+1)h_{n0}=(m+\tfrac{1}{2})\pi\mathrm{i} \tag{27a}$$

$$h'_{n1}=-\tfrac{1}{2}(\mathrm{e}^{2K_1}-1)h'_{n0} \tag{27b}$$

$$h'_{n2}=\tfrac{1}{4}(\mathrm{e}^{2K_1}-1)^2h'_{n0}. \tag{27c}$$

Inspection of (23) indicates that the expansion can be continued to all powers of $(n+1)^{-1}$. Once again we conclude that $h=0$ is a limit point of solutions of (23).

3. Griffiths singularities from decimation

3.1. Decimation in one dimension

Consider an Ising chain with a distribution of bonds assigned according to (2), with $K_2 = \infty$. If n spins are connected by K_2 bonds, they act like a single giant moment. Thus the magnetisation per spin of the system is equal to the magnetisation per spin of a system of giant moments interacting with exchange bond K_1; in other words, our problem has been mapped into an Ising chain with exchange K_1 in a random magnetic field at each spin site, for which the distribution is

$$P(H) = (1-p)^2 \sum_{n=1}^{\infty} np^{n-1}\delta(H - nh). \tag{28}$$

The magnetisation associated with $P(H)$ correctly reduces to (1) when $K_1 = 0$.

For $K_1 \neq 0$ we select a spin A in the chain and trace over all spins except A. This leaves A a free spin in a renormalised magnetic field. The exchange distribution $P(K)$ determines the distribution $P^*(H^*)$ of renormalised fields H^*. If there is no finite value of H^* beyond which $P^*(H^*)$ vanishes and if the contributions to $\bar{m}$ from $H^* \to \infty$ and $H^* \to -\infty$ do not cancel, a Griffiths singularity in the average magnetisation occurs (Grinstein *et al* 1976).

We are unable to write down a useful expression for H^*. We are interested in the analytic properties of $\bar{m}(h)$ for h near zero. Therefore we linearise H^* in h, which is an increasingly good approximation as h becomes arbitrarily small; the factor nh in (28) becomes arbitrarily large as n increases for fixed h (Harris 1975), but for fixed n it can be made arbitrarily small by choosing h close enough to zero. It is straightforward to show that

$$P^*(H^*) = \int \prod_{n=1}^{\infty} (\mathrm{d}H_{n1}\, \mathrm{d}H_{n2} P(H_{n1})P(H_{n2}))\, \mathrm{d}H_A P(H_A)$$

$$\times\, \delta\Big(H - H_A - \sum_{m=1}^{\infty} t_1^m (H_{m1} + H_{m2})\Big) \tag{29}$$

by generating the high-temperature series for H^* in powers of $t_1 = \tanh K_1$ (cf Grinstein *et al* 1976). H_{n1} and H_{n2} refers to the fields on spins which are respectively n bonds to the left and right of spin A. Clearly $P^*(H^*) = 0$ for $H^* < h$ if $t_1 \geqslant 0$ and $h \geqslant 0$; if $t_1 < 0$, P^* has weight at both positive and negative values of H^*.

If $t_1 \geqslant 0$ it can be shown that $P^*(H^*)$ extends to infinity and hence induces a Griffiths singularity. Consider

$$Q(P^*(x)) \equiv \int_x^{\infty} \mathrm{d}H^* P^*(H^*) \tag{30a}$$

$$= \int \prod_{n=1}^{\infty} (\mathrm{d}H_{n_1}\, \mathrm{d}H_{n2} P(H_{n1})P(H_{n2})\, \mathrm{d}H_A P(H_A))$$

$$\times\, \theta\Big(-x + H_A + \sum_{m=1}^{\infty} t_1^m (H_{m1} + H_{m2})\Big) \tag{30b}$$

$$\geqslant \int \mathrm{d}H_A P(H_A)\theta(H_A - x) = Q(P(x)). \tag{30c}$$

So if $Q(P(x))>0$ for arbitrarily large x, $Q(P^*(x))>0$ also. If $t_1<0$ there can be negative contributions to the theta-function from the sums in (30b), and the argument breaks down. Similar results appear to hold on the Bethe lattice.

It is instructive to express this argument more loosely. $P(H)$ consists of equally spaced delta-functions whose weight is np^n. On a 'coarse-grained' scale, $P(H)$ decays like $H\,e^{-\alpha H}$. Thus we can gain insight (Harris 1975) by determining the P^* associated with the distribution

$$P(H)=\tilde{H}^{-1}\,e^{-H/\tilde{H}}\theta(H). \tag{31}$$

The Fourier transform of (31) is

$$P(s)=\int dH\,e^{-isH}P(H)=\frac{1}{i\tilde{H}s+1}. \tag{32}$$

Taking the Fourier transform of (29), we obtain

$$P^*(s)=P(s)\prod_{n=1}^{\infty}(P(st_1^n))^2 \tag{33a}$$

$$=1/(i\tilde{H}s+1)\prod_{n=1}^{\infty}(i\tilde{H}t_1^n s+1)^2. \tag{33b}$$

Inverting the Fourier transform by contour integration and selecting the dominant pole, we find for large positive H^*

$$P^*(H^*)\xrightarrow[H^*\to\infty]{}\frac{e^{-H^*/\tilde{H}}}{\tilde{H}\prod_{n=1}^{\infty}(1-t_1^n)^2}+O(e^{-H^*/(\tilde{H}t_1^m)}), \tag{34}$$

where $m=1$ if $t_1\geqslant 0$ and $m=2$ if $t_1<0$. For negative H^*, $P^*(H^*)$ is zero unless t_1 is negative, in which case

$$P^*(H^*)\underset{H^*\to-\infty}{\sim}e^{-H^*/(\tilde{H}t_1)}+\text{(higher-order terms)}. \tag{35}$$

Substituting (34) and (35) into

$$\bar{m}(\tilde{H})\equiv\int_{-\infty}^{\infty}dH^*P^*(H^*)\tanh H^*, \tag{36}$$

we find that the poles of $\tanh H^*$ on the imaginary axis get smeared down arbitrarily close to the origin, so that an expansion of $\bar{m}$ in powers of $\tilde{H}$ will not converge. From (34) one sees that the amplitude of the leading exponential in $P^*(H^*)$ increases for $t_1>0$ in agreement with (30), and decreases for $t_1<0$; for $t_1<0$ the tails of the distribution near $H=\pm\infty$ do not cancel out the singularity at $\tilde{H}=0$.

The correction term in (34) indicates that $\bar{m}$ is non-analytic not only in $\tilde{H}$ but also in $\tilde{H}t_1^n$. This underscores the riskiness of power-series expansions in t_1 or K_1.

In summary, (30) and (34) suggest that the Griffiths singularity persists when $K_1>0$; (34) suggests that it persists when $K_1<0$.

3.2. Decimation in $d>1$ dimensions

In this section we describe the arguments of § 3.1 in a weakened form which, however, is generalised to an arbitrary number of dimensions. Consider a magnetic field h acting on a d-dimensional random Ising system for which $\bar{m}(H=0)=0$, and whose bonds are distributed according to (2) with $K_2 > K_1$; in other words the temperature is between the critical temperatures of the pure and disordered system and there is no spontaneous long-range order. If one traces over all spins on the lattice except some spin A, h will as before be transformed into a renormalised field H^* acting on A; the magnetisation of A is $m_A = \tanh H^*$. The magnetisation per spin of the random system is calculated by taking the configurational average of m_A over $P(K)$, which is equivalent to determining $P^*(H^*)$ in (20) and (36).

As in § 3, we take h small and calculate H^* to first order in h:

$$H^* = \tfrac{1}{2}\ln[(\operatorname*{tr}_{s_A=1} e^{-\beta\mathcal{H}})/(\operatorname*{tr}_{s_A=-1} e^{-\beta\mathcal{H}})] \tag{37a}$$

$$\simeq h\sum_j \langle s_A s_j\rangle \tag{37b}$$

$$\langle X\rangle \equiv (\operatorname{tr} X\, e^{-\beta\mathcal{H}})/(\operatorname{tr} e^{-\beta\mathcal{H}}). \tag{37c}$$

After writing

$$e^{-\beta\mathcal{H}} = \left[\prod_{ij}(1+s_i s_j \tanh K_{ij})\cosh K_{ij}\right]\left[\prod_i (1+s_i\tanh h)\cosh h\right], \tag{38}$$

it is especially easy to see from (37a) that on a Bethe lattice the coefficient of $t_1^n t_2^m$ in the expansion for H^* is positive. Then H^* increases as t_1 increases. Since Griffiths singularities exist for $t_1 = 0$, $P^*(H^*)$ extends to ∞ for $t_1 > 0$. But since H^* increases with K_1, $P^*(H^*)$ must extend to ∞ for $t_1 > 0$ as well. Thus Griffiths singularities exist for a binary mixture of ferromagnetic bonds on a Bethe lattice.

We emphasise that (37b) does not converge below the ordering temperature of the random system. In fact the convergence of the expansion of H^* in the t_i's is doubtful even in the temperature range we consider (cf Griffiths 1969), and the argument above is only suggestive.

However, one expects that $\chi_{Aj} \equiv \langle s_A s_j\rangle$ in (37b) usually increases with K_1 for fixed p and K_2, i.e. as diluted bonds are turned into weak ferromagnetic ones: the configurational average of $\Sigma_j \chi_{Aj}$ is the susceptibility $\bar{\chi}$. Until it diverges when the critical point is reached, $\bar{\chi}$ should increase with K_1. So if H^* usually increases with K_1, the existence of Griffiths singularities for $K_1 = 0$ implies their existence for $K_1 > 0$ as well.

4. Discussion

In this paper we have investigated the existence of a delicate singularity in the magnetisation of the random Ising chain. Not finding a rigorous argument, we have given two rather different non-rigorous ones instead. The agreement between two dissimilar arguments strengthens our confidence in the validity of our result.

In § 2 an expansion of $\nu(R)$ in powers of p and an iterative solution to (10) were given, both without regard to convergence. The basic premise of § 2 was that a Griffiths singularity in the non-dilute system would be maintained or suppressed by the same terms in $\nu_n(R)$, equation (18), which gave rise to the singularity in the dilute case. Thus $\nu^{(1)}(R)$ can be regarded as an approximate solution to the integral equation (7) which reduces to the exact solution when $K_1 = 0$. Our conclusions from $\nu^{(1)}(R)$ were checked by expanding the solutions to equation (18) in powers of K_1 and of $(n+1)^{-1}$.

In § 3.1 we studied the magnetisation of the chain by treating groups of spins locked together by infinitely strong bonds as giant moments; our system was mapped into an ordinary Ising chain in a random magnetic field. By tracing over all spins in the system except some spin A, we generated a renormalised magnetic field H^* acting on A. Since the Griffiths singularity occurs for small h, the formula for H^* was linearised in h; the region of the tail of $P^*(H^*)$ in which this is a bad approximation can be driven arbitrarily far from the origin by making h arbitrarily small. A Griffiths singularity was found for $K_1 > 0$; coarse-graining the distribution (18) for the dilute chain led to a Griffiths singularity for $K_1 < 0$ as well. Decimation also was used in § 3.2 to study a binary mixture of ferromagnetic bonds for $d > 1$. Perturbation theory suggests the existence of a Griffiths singularity, as do the general properties of equation (37*b*) for H^*, but the case made is not compelling.

In short, our arguments for the existence of a Griffiths singularity seem highly plausible when $d = 1, K_2 = \infty$ and $K_1 > 0$; plausible when $d = 1, K_2 = \infty$ and $K_1 < 0$; and suggestive but inconclusive when $d > 1$, $K_2 > K_1 > 0$.

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References

Bakri M M and Stauffer D 1976 *Phys. Rev.* B **14** 4215
Chalupa J, Grinstein G and Wortis M 1976 *J. Chem. Phys.* **11** 4512
Domb C 1974a *J. Phys. C: Solid St. Phys.* **7** 2677
—— 1974b *J. Phys. C: Solid St. Phys.* **7** 4552
Fan C and McCoy B M 1969 *Phys. Rev.* **182** 614
Fernández J F 1977 *Phys. Rev.* B **16** 5125
Griffiths R B 1969 *Phys. Rev. Lett.* **23** 17
Grinstein G, Berker A N, Chalupa J and Wortis M 1976 *Phys. Rev. Lett.* **36** 1508
Harris A B 1975 *Phys. Rev.* B **12** 203
Imry Y 1977 *Phys. Rev.* B **15** 4448
Landau D P and Blume M 1976 *Phys. Rev.* B **13** 287
—— 1977 *Phys. Rev.* B **16** 598E
Lehman G W and McTague J P 1968 *J. Chem. Phys.* **49** 3170
Lieb E H and Mattis D C 1966 *Mathematical Physics in One Dimension* (New York: Academic) pp 119–24
Matsubara F, Yoshimura K and Katsura S 1973 *Can. J. Phys.* **51** 1053
de Menezes L C, Rauh A and Salinas S R 1977 *Phys. Rev.* B **15** 3485
Morgenstern I, Binder K and Baumgärtner A 1978 *J. Chem. Phys.* **69** 253
Rauh A 1976a *Phys. Rev.* B **13** 2080
—— 1976b *Phys. Rev.* B **14** 1997
Schwartz M 1978 *Phys. Rev.* B **18** 2364
Vedenov A A and Dykhne A M 1968 *Zh. Eksp. Teor. Fiz.* **55** 357 (Engl transl. 1969 *Sov. Phys.-JETP* **28** 187)
Vilenkin A 1978 *Phys. Rev.* B **18** 1474
Wortis M 1974 *Phys. Rev.* B **10** 4665